GROUPS, ALGEBRAS AND MEANS

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Abstract. The paper contains loose notes on the subject of amenability of discrete groups. The origin of these notes is a series of lectures the author gave at the "Operator Algebras and Quantum Groups" seminar at the Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw.

Throughout these notes Γ will be a discrete group with unit element denoted by e. We will consider various algebras associated with Γ. In particular the group algebra $\mathbb{C}[\Gamma]$ and the convolution algebra $\ell^1(\Gamma)$. Note that $\mathbb{C}[\Gamma]$ is a dense unital $*$-subalgebra of $\ell^1(\Gamma)$. The convolution of functions on Γ will be denoted by the symbol "$*$":

$$(f*g)(t)=\sum_{s\in\Gamma}f(s)g(s^{-1}t).$$

The algebra $\ell^1(\Gamma)$ carries an isometric involution

$$f^*(t)=\overline{f(t^{-1})}$$

for $f\in\ell^1(\Gamma)$ and all $t\in\Gamma$.

For $t\in\Gamma$ the symbol δ_t will denote the function $\Gamma\to\mathbb{C}$ whose value at t is 1 and which is zero elsewhere. The functions $\{\delta_t\}_{t\in\Gamma}$ will be treated as elements of $\mathbb{C}[\Gamma]$, $\ell^1(\Gamma)$ or $\ell^2(\Gamma)$ depending on what we need.

Throughout the notes Hilbert space scalar products will be linear in the *second* variable. We will use the notation $(\phi|\psi)$ for the scalar product of vectors ϕ and ψ and if T is an operator on the Hilbert space in question we will at times write $(\phi|T|\psi)$ for the "matrix element" of T which is the same thing as $(\phi|T\psi)$.

For a Banach space X the symbol X_1 will denote the closed unit ball of X. Moreover, sometimes we will write Y_1 for an intersection of a subset Y of X with X_1.

1. Representations of Γ and $\ell^1(\Gamma)$

By $\operatorname{Rep}(\Gamma)$ we will denote the class of all unitary representations of Γ. For each $U\in\operatorname{Rep}(\Gamma)$ the Hilbert space on which U acts will be denoted by H_U. For $f\in\ell^1(\Gamma)$ and $U\in\operatorname{Rep}(\Gamma)$ we define $U_f\in\mathrm{B}(H_U)$

$$U_f=\sum_{t\in\Gamma}f(t)U_t$$

So defined mapping $\ell^1(\Gamma)\ni f\mapsto U_f\in\mathrm{B}(H_U)$ is a unital $*$-homomorphism.

For any $t\in\Gamma$ we denote by δ_t the characteristic function of $\{t\}$.

Proposition 1.1. *There is a bijection between unital $*$-representations of $\ell^1(\Gamma)$ on Hilbert spaces and unitary representations of Γ.*

In one direction we already constructed a representation of $\ell^1(\Gamma)$ from an element of $\operatorname{Rep}(\Gamma)$. Conversely, given a unital $*$-representation $\pi:\ell^1(\Gamma)\to\mathrm{B}(H)$ for some Hilbert space H we define $U_t=\pi(\delta_t)$. Clearly this is a unitary representation of Γ and for each $f\in\ell^1(\Gamma)$ we have

$$U_f=\pi(f).$$

2. Group C*-algebra

For any element $f \in \ell^1(\Gamma)$ we define[i]

$$\|f\|_{\mathrm{C}^*(\Gamma)} = \sup_{U \in \mathrm{Rep}(\Gamma)} \|U_f\|_{\mathrm{B}(H_U)}.$$

This is always finite because for each $f \in \ell^1(\Gamma)$ and $U \in \mathrm{Rep}(\Gamma)$

$$\|U_f\|_{\mathrm{B}(H_U)} = \Bigl\|\sum_{t\in\Gamma} f(t)U_t\Bigr\|_{\mathrm{B}(H_U)} \leq \sum_{t\in\Gamma}\bigl|f(t)\bigr| = \|f\|_1.$$

Let us define the *regular representation* $\lambda : \Gamma \to \mathrm{B}\bigl(\ell^2(\Gamma)\bigr)$ in which an element $t \in \Gamma$ is mapped to the unitary operator $\lambda_t \in \mathrm{B}\bigl(\ell^2(\Gamma)\bigr)$ acting as

$$(\lambda_t\psi)(s) = \psi(t^{-1}s)$$

for any $\psi \in \ell^2(\Gamma)$ and $s \in \Gamma$. By considering the regular representation we see that $\|\cdot\|_{\mathrm{C}^*(\Gamma)}$ is a norm, since for any $f \in \ell^1(\Gamma)$ we have

$$\|f\|_{\mathrm{C}^*(\Gamma)} \geq \|\lambda_f\|_{\mathrm{B}(\ell^2(\Gamma))} \geq \|\lambda_f\delta_e\|_2 = \|f\|_2$$

which is zero if and only if $f = 0$.

The completion of $\ell^1(\Gamma)$ with respect to $\|\cdot\|_{\mathrm{C}^*(\Gamma)}$ is denoted by $\mathrm{C}^*(\Gamma)$ and it is a unital C*-algebra. Any representation of $\mathrm{C}^*(\Gamma)$ gives by restriction a unital $*$-representation of $\ell^1(\Gamma)$ and, by construction, for any unital $*$-representation π of $\ell^1(\Gamma)$ on a Hilbert space H there exists a unique representation of $\mathrm{C}^*(\Gamma)$ whose restriction to $\ell^1(\Gamma)$ is π. It follows from Proposition 1.1 that representations of $\mathrm{C}^*(\Gamma)$ are in bijective correspondence with unitary representations of Γ.

3. Positive definite functions

Definition 3.1. A function $\varphi : \Gamma \to \mathbb{C}$ is *positive definite* if for any $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \Gamma$ the matrix

$$\begin{bmatrix} \varphi(t_1^{-1}t_1) & \varphi(t_1^{-1}t_2) & \cdots & \varphi(t_1^{-1}t_n) \\ \varphi(t_2^{-1}t_1) & \varphi(t_2^{-1}t_2) & \cdots & \varphi(t_2^{-1}t_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(t_n^{-1}t_1) & \varphi(t_n^{-1}t_2) & \cdots & \varphi(t_n^{-1}t_n) \end{bmatrix}$$

is positive, i.e. for any $z_1, \dots, z_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} \overline{z_i}\varphi(t_i^{-1}t_j)z_j \geq 0.$$

Proposition 3.2. *Let φ be a positive definite function on Γ then*

1. $\varphi(t^{-1}) = \overline{\varphi(t)}$ *for all* $t \in \Gamma$,
2. $\bigl|\varphi(t)\bigr| \leq \varphi(e)$ *for all* $t \in \Gamma$.

Proof. For any $t \in \Gamma$ the matrix

$$\begin{bmatrix} \varphi(e) & \varphi(t) \\ \varphi(t^{-1}) & \varphi(e) \end{bmatrix}$$

is positive. Therefore it must be selfadjoint (so that $\varphi(e) \in \mathbb{R}$ and $\varphi(t^{-1}) = \overline{\varphi(t)}$). Choosing $z_1 = 1$ and z_2 such that $|z_2| = 1$ and $z_2\varphi(t) = -\bigl|\varphi(t)\bigr|$ we get

$$\begin{aligned} 0 &\leq \overline{z_1}\varphi(e^{-1}e)z_2 + \overline{z_1}\varphi(e^{-1}t)z_2 + \overline{z_2}\varphi(t^{-1}e)z_1 + \overline{z_2}\varphi(t^{-1}t)z_2 \\ &= \bigl(|z_1|^2 + |z_2|^2\bigr)\varphi(e) + 2\Re\overline{z_1}z_2\varphi(t) \end{aligned}$$

(because $\varphi(t^{-1}) = \overline{\varphi(t)}$) and (2) follows. ☐

[i] There is no need to worry about taking supremum over a class because we are in fact taking a supremum of an image of the class $\mathrm{Rep}(\Gamma)$ inside the *set* $\mathbb{R}$.

In particular all positive definite functions are bounded and their ℓ^∞-norm is attained at $e \in \Gamma$.

Also notice that the complex conjugate of a positive definite function is positive definite and a pointwise product of positive definite functions if positive definite. This is because the complex conjugate of a positive matrix is positive and entry-wise product of positive matrices is positive:

Proposition 3.3. *Let*

$$A = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,n} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n,1} & \beta_{n,2} & \cdots & \beta_{n,n} \end{bmatrix}$$

be positive matrices and let

$$C = \begin{bmatrix} \alpha_{1,1}\beta_{1,1} & \alpha_{1,2}\beta_{1,2} & \cdots & \alpha_{1,n}\beta_{1,n} \\ \alpha_{2,1}\beta_{2,1} & \alpha_{2,2}\beta_{2,2} & \cdots & \alpha_{2,n}\beta_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,1}\beta_{n,1} & \alpha_{n,2}\beta_{n,2} & \cdots & \alpha_{n,n}\beta_{n,n} \end{bmatrix}$$

Then C is positive.

Proof. Let us fix $z_1, \ldots, z_n \in \mathbb{C}$. Since B is positive, there is a matrix

$$D = \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,n} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n,1} & \delta_{n,2} & \cdots & \delta_{n,n} \end{bmatrix}$$

such that $B = D^*D$. We have

$$\sum_{i,j=1}^{n} \overline{z_i}\alpha_{i,j}\beta_{i,j}z_j = \sum_{i,j=1}^{n} \overline{z_i}\alpha_{i,j}\left(\sum_{k=1}^{n} \overline{\delta_{k,i}}\delta_{k,j}\right)z_j = \sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} \overline{(z_i\delta_{k,i})}\,\alpha_{i,j}\,(z_j\delta_{k,j})\right)$$

which is positive as each summand of the outer sum is positive. □

Example 3.4. Let $U \in \mathrm{Rep}(\Gamma)$ and choose $\xi \in H_U$ then the function

$$\varphi : \Gamma \ni t \longmapsto (\xi|U_t|\xi) \in \mathbb{C}$$

is positive definite and $\varphi(e) = \|\xi\|^2$. Indeed if $t_1, \ldots, t_n \in \Gamma$ and $z_1, \ldots, z_n \in \mathbb{C}$ then

$$\begin{aligned} \sum_{i,j=1}^{n} \overline{z_i}\varphi(t_i^{-1}t_j)z_j &= \sum_{i,j=1}^{n} \overline{z_i}\left(\xi\middle|U_{t_i}^{-1}U_{t_j}\middle|\xi\right)z_j \\ &= \sum_{i,j=1}^{n}\left(z_iU_{t_i}\xi\middle|z_jU_{t_j}\xi\right) \\ &= \left\|\sum_{i=1}^{n} z_iU_{t_i}\xi\right\|^2 \geq 0 \end{aligned}$$

Any positive definite function φ on Γ gives a linear functional on $\ell^1(\Gamma)$ (being an element of $\ell^\infty(\Gamma)$). We will denote this functional by ω_φ:

$$\omega_\varphi(f) = \sum_{t\in\Gamma} \varphi(t)f(t)$$

for $f \in \ell^1(\Gamma)$.

Proposition 3.5. *For any positive definite φ the functional ω_φ is positive in the sense that for any $f \in \ell^1(\Gamma)$ we have*

$$\omega_\varphi(f^* * f) \geq 0. \tag{1}$$

Proof. For $f \in \ell^1(\Gamma)$ let $(f_n)_{n\in\mathbb{N}}$ be a sequence of finitely supported functions on Γ converging to f in ℓ^1. Since ω_φ is continuous ($\|\omega_\varphi\| = \|\varphi\|_\infty = \varphi(e)$) we have

$$\omega_\varphi(f^* * f) = \lim_{n\to\infty} \varphi(f_n^* * f_n).$$

Therefore it is enough to prove (1) for a finitely supported f. To that end let

$$f = \sum_{i=1}^{n} z_{t_i}\delta_{t_i}.$$

We have

$$\begin{aligned}
\omega_\varphi(f^* * f) &= \omega_\varphi\biggl(\biggl(\sum_{i=1}^{n} \overline{z_{t_i}}\delta_{t_i^{-1}}\biggr) * \biggl(\sum_{j=1}^{n} z_{t_j}\delta_{t_j}\biggr)\biggr)\\
&= \sum_{i,j=1}^{n} \overline{z_{t_i}}\omega_\varphi\bigl(\delta_{t_i^{-1}} * \delta_{t_j}\bigr)z_{t_j}\\
&= \sum_{i,j=1}^{n} \overline{z_{t_i}}\omega_\varphi\bigl(\delta_{t_i^{-1}t_j}\bigr)z_{t_j} = \sum_{i,j=1}^{n} \overline{z_{t_i}}\varphi(t_i^{-1}t_j)z_{t_j} \geq 0.
\end{aligned}$$

□

Before we continue we need to turn our attention to the following fact: we know that for a positive definite function φ the functional ω_φ is positive and attains its norm on δ_e. However this is true for all positive functionals, not only those of the form ω_φ (in fact the following proof works for a positive linear functional on any $*$-algebra). To see this let ω be a positive functional on $\ell^1(\Gamma)$ and fixing $f, g \in \ell^1(\Gamma)$ consider the polynomial

$$P(t) = \omega\bigl((f + tg)^* * (f + tg)\bigr).$$

Then $P(t) \geq 0$ for all $t \in \mathbb{R}$. In particular the coefficient of t must be real:

$$\omega(f^* * g) + \omega(g^* * f) \in \mathbb{R}.$$

For $g = \delta_e$ we obtain $\omega(f^*) + \omega(f) \in \mathbb{R}$ and if moreover $f = f^*$ we get $\omega(f) \in \mathbb{R}$. This means that ω is a selfadjoint functional:

$$\begin{aligned}
\omega(h^*) &= \omega\Bigl(\bigl(\tfrac{h+h^*}{2} + \mathrm{i}\tfrac{h-h^*}{2\mathrm{i}}\bigr)^*\Bigr)\\
&= \omega\bigl(\tfrac{h+h^*}{2} - \mathrm{i}\tfrac{h-h^*}{2\mathrm{i}}\bigr)\\
&= \omega\bigl(\tfrac{h+h^*}{2}\bigr) - \mathrm{i}\omega\bigl(\tfrac{h-h^*}{2\mathrm{i}}\bigr)\\
&= \overline{\omega\bigl(\tfrac{h+h^*}{2}\bigr) + \mathrm{i}\omega\bigl(\tfrac{h-h^*}{2\mathrm{i}}\bigr)} = \overline{\omega(h)}
\end{aligned}$$

for any $h \in \ell^1(\Gamma)$.

This shows that

$$P(t) = \omega(g^* * g)t^2 + 2\Re\,\omega(f^* * g)t + \omega(g^* * g)$$

and the standard method yields the Schwarz inequality

$$\bigl|\omega(f^* * g)\bigr|^2 \leq \omega(f^* * f)\omega(g^* * g).$$

To see that $\|\omega\| = \omega(\delta_e)$ we note first that clearly $\|\omega\| \geq \omega(\delta_e)$. On the other hand the Schwarz inequality shows that for any $f \in \ell^1(\Gamma)$

$$\bigl|\omega(f)\bigr|^2 = \bigl|\omega(\delta_e^* * f)\bigr|^2 \leq \omega(\delta_e)\omega(f^* * f) \leq \omega(\delta_e)\|\omega\|\|f\|_1^2.$$

Taking $\sup\limits_{\|f\|_1=1}$ on both sides gives

$$\|\omega\|^2 \leq \omega(\delta_e)\|\omega\|$$

which shows that $\|\omega\| \leq \omega(\delta_e)$.

In particular for any $f,g\in\ell^1(\Gamma)$ we have

$$\begin{aligned}0\leq\omega(g^**f^**f*g)&\leq\big\|\omega(g^**\cdot*g)\big\|\|f^**f\|_1\\&=\omega(g^**\delta_e*g)\|f^**f\|_1=\omega(g^**g)\|f^**f\|_1.\end{aligned}$$

Thus if $\omega(g^**g)=0$ then $\omega\big((f*g)^**(f*g)\big)=0$ and it follows that the subspace

$$\mathcal{N}_\omega=\big\{g\in\ell^1(\Gamma)\,\big|\,\omega(g^**g)=0\big\}$$

is a left ideal of $\ell^1(\Gamma)$. (It follows from the Schwarz inequality that this is a vector subspace of $\ell^1(\Gamma)$.)

Proposition 3.6. *Let φ be a positive definite function on Γ. Then there exists $U\in\mathrm{Rep}(\Gamma)$ and $\xi\in H_U$ such that*

$$\varphi(t)=(\xi|U_t|\xi)$$

for all $t\in\Gamma$.

Proof. Let $\mathcal{N}=\mathcal{N}_{\omega_\varphi}$ as above, i.e

$$\mathcal{N}=\big\{f\in\ell^1(\Gamma)\,\big|\,\omega_\varphi(f^**f)=0\big\}.$$

Since $\mathcal{N}$ is an ideal in $\ell^1(\Gamma)$ we have a representation π of $\ell^1(\Gamma)$ on $\mathcal{H}=\ell^1(\Gamma)/\mathcal{N}$ by

$$\pi(f)(h+\mathcal{N})=f*h+\mathcal{N}.$$

Now $\mathcal{H}$ is a pre-Hilbert space with scalar product

$$(f+\mathcal{N}|g+\mathcal{N})=\omega_\varphi(f^**g).$$

and each $\pi(f)$ is bounded:

$$\begin{aligned}\big\|\pi(f)(h+\mathcal{N})\big\|^2&=(f*h+\mathcal{N}|f*h+\mathcal{N})\\&=\omega_\varphi\big((f*h)^**(f*h)\big)\\&\leq\omega_\varphi(h^**h)\|f^**f\|_1\\&\leq\omega_\varphi(h^**h)\|f\|_1^2=\|h+\mathcal{N}\|^2\|f\|_1^2,\end{aligned}$$

so that $\big\|\pi(f)\big\|\leq\|f\|_1$.

Upon completing $\mathcal{H}$ to a Hilbert space H we obtain a unital $*$-representation π of $\ell^1(\Gamma)$ on H, since

$$\begin{aligned}(h+\mathcal{N}|\pi(f)(g+\mathcal{N}))&=(h+\mathcal{N}|f*g+\mathcal{N})\\&=\omega_\varphi\big(h^**f*g\big)\\&=\omega_\varphi\big((f^**h)^**g\big)=\left(\pi(f^*)(h+\mathcal{N})|g+\mathcal{N}\right).\end{aligned}$$

Moreover the image ξ of δ_e in the obvious map $\ell^1(\Gamma)\to H$ is a cyclic vector for π and we have

$$\omega_\varphi(f)=\left(\xi|\pi(f)|\xi\right).$$

Recall that $\varphi(t)=\omega_\varphi(\delta_t)$. This means that

$$\varphi(t)=\left(\xi|U_t|\xi\right),$$

where $U\in\mathrm{Rep}(\Gamma)$ is defined as $U_t=\pi(\delta_t)$ (cf. Proposition 1.1). □

Corollary 3.7. *Let φ be a positive definite function on Γ. Then the positive functional ω_φ on $\ell^1(\Gamma)$ is continuous for the norm $\|\cdot\|_{\mathrm{C}^*(\Gamma)}$.*

Proof. We have

$$\big|\omega_\varphi(f)\big|=\big|(\xi|\pi(f)|\xi)\big|\leq\|\xi\|^2\big\|\pi(f)\big\|\leq\|\xi\|^2\|f\|_{\mathrm{C}^*(\Gamma)}$$

for any $f\in\ell^1(\Gamma)$. □

Therefore any positive definite function φ on Γ defines a positive functional ω_φ on $\mathrm{C}^*(\Gamma)$.

4. Positive definite functions and states on $\mathrm{C}^*(\Gamma)$

Let us denote by $\mathcal{P}_1(\Gamma)$ the set of those positive definite functions on Γ which have value 1 at the neutral element e. Clearly for any $\varphi\in\mathcal{P}_1(\Gamma)$ the positive functional ω_φ on $\mathrm{C}^*(\Gamma)$ is a state.[ii] We will denote the state space of $\mathrm{C}^*(\Gamma)$ by the symbol $\mathcal{S}\bigl(\mathrm{C}^*(\Gamma)\bigr)$.

Proposition 4.1. *The mapping*

$$\mathcal{P}_1(\Gamma)\ni\varphi\longmapsto\omega_\varphi\in\mathcal{S}\bigl(\mathrm{C}^*(\Gamma)\bigr) \tag{2}$$

is a bijection and a homeomorphism for the topology of pointwise convergence on $\mathcal{P}_1(\Gamma)$ and weak* topology on $\mathcal{S}\bigl(\mathrm{C}^*(\Gamma)\bigr)$.

Proof. For $\omega\in\mathcal{S}\bigl(\mathrm{C}^*(\Gamma)\bigr)$ let π_ω be the GNS representation of $\mathrm{C}^*(\Gamma)$ with cyclic vector Ω_ω. Define

$$\varphi_\omega:\Gamma\ni t\longmapsto(\Omega_\omega|\pi_\omega(\delta_t)|\Omega_\omega)\,.$$

Clearly φ_ω is a positive definite function. We have for any $f\in\ell^1(\Gamma)$

$$\begin{aligned}\omega_{\varphi_\omega}(f)&=\sum_{t\in\Gamma}\varphi_\omega(t)f(t)\\&=\sum_{t\in\Gamma}(\Omega_\omega|\pi_\omega(\delta_t)|\Omega_\omega)\,f(t)\\&=\left(\Omega_\omega\middle|\sum_{t\in\Gamma}f(t)\pi_\omega(\delta_t)\middle|\Omega_\omega\right)\\&=(\Omega_\omega|\pi_\omega(f)|\Omega_\omega)=\omega(f).\end{aligned}$$

Similarly for a fixed $\varphi\in\mathcal{P}_1(\Gamma)$ and any $t\in\Gamma$ we have

$$\varphi_{\omega_\varphi}(t)=\bigl(\Omega_{\omega_\varphi}\big|\pi_{\omega_\varphi}(\delta_t)\big|\Omega_{\omega_\varphi}\bigr)=\omega_\varphi(\delta_t)=\varphi(t),$$

which shows that (2) is a bijection.

If (ω_i) is a net of states on $\mathrm{C}^*(\Gamma)$ converging to a state ω in the weak* topology then

$$\omega_i(\delta_t)\longrightarrow\omega(\delta_t)$$

for any $t\in\Gamma$, i.e. $\varphi_{\omega_i}\xrightarrow{\text{pointwise}}\varphi_\omega$.

Conversely if (φ_i) is a net in $\mathcal{P}_1(\Gamma)$ converging pointwise to an element $\varphi\in\mathcal{P}_1(\Gamma)$ then

$$\omega_{\varphi_i}(\delta_t)\longrightarrow\omega_\varphi(\delta_t)$$

for any t and thus

$$\omega_{\varphi_i}(x)\longrightarrow\omega_\varphi(x)$$

for any $x\in\mathbb{C}[\Gamma]$. Given $y\in\mathrm{C}^*(\Gamma)$ we can for any $\varepsilon>0$ find $x_\varepsilon\in\mathbb{C}[\Gamma]$ such that

$$\|y-x_\varepsilon\|_{\mathrm{C}^*(\Gamma)}<\tfrac{\varepsilon}{3}.$$

Moreover there exists i_ε such that for all $i\succcurlyeq i_\varepsilon$ we have

$$\bigl|\omega_{\varphi_i}(x_\varepsilon)-\omega_\varphi(x_\varepsilon)\bigr|<\tfrac{\varepsilon}{3}.$$

Since all ω_{φ_i} and ω_φ have norm 1 on $\mathrm{C}^*(\Gamma)$, it follows that

$$\bigl|\omega_{\varphi_i}(y)-\omega_\varphi(y)\bigr|\leq\bigl|\omega_{\varphi_i}(y)-\omega_{\varphi_i}(x_\varepsilon)\bigr|+\bigl|\omega_{\varphi_i}(x_\varepsilon)-\omega_{\varphi_i}(x_\varepsilon)\bigr|+\bigl|\omega_\varphi(x_\varepsilon)-\omega_\varphi(y)\bigr|<\varepsilon$$

for all $i\succcurlyeq i_\varepsilon$. □

[ii] A *state* on a C^*-algebra is a positive linear functional of norm 1. A positive functional always attains its norm on the unit.

5. Amenability

We define left and right translation operations on functions on Γ: let f be any function on Γ, for $t \in \Gamma$ we define

$$L_t f(s) = f(t^{-1}s), \qquad R_x f(s) = f(sx)$$

for all $s \in \Gamma$.

Definition 5.1. Γ is *amenable* if there exists a state m on $\ell^\infty(\Gamma)$ such that

$$m(L_t f) = m(f)$$

for all $f \in \ell^\infty(\Gamma)$ and $t \in \Gamma$. Such a state m is called a *left invariant mean* on Γ.

The next proposition shows how invariance of the mean can be expressed without using points of the set Γ.

Proposition 5.2. *Let m be a left invariant mean on Γ. Then for any $\phi \in \ell^1(\Gamma)$ such that $\sum\limits_{s\in\Gamma} \phi(s) = 1$ we have*

$$m(\phi * f) = m(f)$$

for all $f \in \ell^\infty(\Gamma)$.

Note that if $f \in \ell^\infty(\Gamma)$ and $\phi \in \ell^1(\Gamma)$ then the convolution product $\phi * f$ is well defined because for each $t \in \Gamma$ the sum

$$\sum_{s\in\Gamma} \phi(s) f(s^{-1}t) = \sum_{s\in\Gamma} \phi(s^{-1}) f(st)$$

converges. Moreover $\phi * f \in \ell^\infty(\Gamma)$ and

$$\|\phi * f\|_\infty = \sup_{t\in\Gamma} \Big| \sum_{s\in\Gamma} \phi(s) f(s^{-1}t) \Big| \le \sup_{t\in\Gamma} \sum_{s\in\Gamma} \big|\phi(s)\big| \big|f(s^{-1}t)\big| \le \|\phi\|_1 \|f\|_\infty.$$

Proof of Proposition 5.2. Take $f \in \ell^\infty(\Gamma)$ and $\phi \in \ell^1(\Gamma)$. We have

$$\big((L_x\phi) * f\big)(t) = \sum_{s\in\Gamma} \phi(x^{-1}s) f(s^{-1}t) = \sum_{s\in\Gamma} \phi(s) f(s^{-1}x^{-1}t) = \big(L_x(\phi * f)\big)(t).$$

Therefore the invariance of m gives

$$m\big((L_x\phi) * f\big) = m\big(L_x(\phi * f)\big) = m(\phi * f).$$

Therefore the map

$$\phi \longmapsto m(\phi * f)$$

is a linear functional on $\ell^1(\Gamma)$ which is bounded:

$$\big|m(\phi * f)\big| \le \|\phi * f\|_\infty \le \|\phi\|_1 \|f\|_\infty$$

and left invariant. It is easy to see that the only such functionals are given by constant functions in $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$. Therefore there exists a constant $k(f)$ such that

$$m(\phi * f) = k(f) \sum_{s\in\Gamma} \phi(s).$$

For ϕ such that $\sum\limits_{s\in\Gamma} \phi(s) = 1$ we get

$$m(\phi * f) = k(f).$$

In particular we can take $\phi = \delta_e$, which gives

$$k(f) = m(\delta_e * f) = m(f).$$

$\square$

Since invariance of m follows from the condition in Proposition 5.2 (by taking $\phi = \delta_t$), we see that m is invariant if and only if it satisfies this condition.

Example 5.3. Let us construct a mean on the group $\mathbb{Z}$. Consider the subspace c of convergent sequences inside $\ell^\infty(\mathbb{N})$. Let $\mathcal{L}$ be an extension to $\ell^\infty(\mathbb{N})$ of the (norm 1) functional $\lim\colon c \to \mathbb{C}$. Now let T be the operator $\ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{N})$ defined as

$$(Tf)(n) = \tfrac{1}{2n+1} \sum_{k=-n} nf(k).$$

Then $m = \mathcal{L}\circ T$ is a linear functional on $\ell^\infty(\mathbb{Z})$ which is positive and has value 1 on the constant function 1. Therefore it is a state. Moreover m is translation-invariant. This is because for any $f \in \ell^\infty(\mathbb{Z})$ and any $t \in \mathbb{Z}$ the function $T(f - L_t f)$ belongs to the subspace c_0 of sequences converging to 0. Indeed, for $\in \mathbb{Z}$ we have

$$\begin{aligned}\bigl|T(f - L_t f)(n)\bigr| &= \left|\tfrac{1}{2n+1} \sum_{k=-n}^{n} f(k) - \tfrac{1}{2n+1} \sum_{l=-n}^{n} f(l-t)\right| \\ &= \tfrac{1}{2n+1}\left|\sum_{k=-n}^{n} f(k) - \sum_{l=-n-t}^{n-t} f(l)\right|.\end{aligned}$$

In the two sums in the last expression many elements cancel. In fact one can see that there are exactly $2|t|$ terms which do not cancel, therefore

$$\bigl|T(f - L_t f)(n)\bigr| \leq \tfrac{2|t|}{2n+1}\|f\|_\infty$$

which converges to 0 as $n \to \infty$.

Example 5.3 shows how highly non-unique are means on amenable groups. By an application of the Markov-Kakutani fixed point theorem one can show that any abelian group is amenable.

Example 5.4. Let us see that the free group on two generators, $\mathbb{F}_2$, is not amenable. When a group Γ is amenable and m is an invariant mean on Γ we can use m to define a translation-invariant finitely additive probability measure μ on Γ by putting $\mu(E) = m(\chi_E)$ for any $E \subset \Gamma$. We will show that such a finitely additive invariant probability measure does not exist on $\mathbb{F}_2$.

Denote the generators of $\mathbb{F}_2$ by a and b and let

$$\begin{aligned}A_+ &= \bigl\{\text{reduced words beginning with } a\bigr\},\\ B_+ &= \bigl\{\text{reduced words beginning with } b\bigr\},\\ A_- &= \bigl\{\text{reduced words beginning with } a^{-1}\bigr\},\\ B_- &= \bigl\{\text{reduced words beginning with } b^{-1}\bigr\}.\end{aligned}$$

The group $\mathbb{F}_2$ is a disjoint union of $\{e\}$ and the sets A_+, B_+, A_- and B_-. Now assume that there exists a finitely additive translation-invariant probability measure μ defined on all subsets of $\mathbb{F}_2$ Note that aA_- consists of all reduced words which do not begin with a. Thus

$$\mathbb{F}_2 = A_+ \cup aA_-$$

(disjoint union). Therefore $1 = \mu(\mathbb{F}_2) = \mu(A_+) + \mu(aA_-) = \mu(A_+) + \mu(A_-)$. Now since

$$bA_+ \subset B_+ \subset aA_-$$

we have $\mu(A_+) = \mu(bA_+) \leq \mu(aA_-) = \mu(A_-)$ and similarly from

$$bA_- \subset B_+ \subset a^{-1}A_+$$

we infer $\mu(A_-) = \mu(bA_-) \leq \mu(a^{-1}A_+) = \mu(A_+)$. Thus $\mu(A_-) = \mu(A_+) = \frac{1}{2}$. But this implies that $\mu(B_+) = \mu(B_-) = 0$, while

$$\mu(B_+) \geq \mu(bA_+) = \mu(A_+) = \tfrac{1}{2}.$$

This contradiction shows that $\mathbb{F}_2$ is not amenable.

6. Permanence properties

Theorem 6.1.

(1) *If Γ is amenable and $\Gamma_0 \subset \Gamma$ is a subgroup then Γ_0 is amenable.*
(2) *If Γ is amenable and Φ is a homomorphism form Γ onto a group Γ_0 then Γ_0 is amenable.*
(3) *If Γ has a normal subgroup Γ_0 such that Γ_0 and Γ/Γ_0 are amenable then Γ is amenable.*
(4) *If Γ has a directed family of subgroups (Γ_α) such that $\Gamma = \bigcup \Gamma_\alpha$ and each Γ_α is amenable then Γ is amenable.*

Proof. Ad (1). Let m be a left invariant mean on Γ and let (x_i) be a system of representatives of right cosets of Γ_0. We have

$$\Gamma = \bigsqcup \Gamma_0 x_i.$$

Define $\Lambda : \ell^\infty(\Gamma_0) \to \ell^\infty(\Gamma)$ by

$$(\Lambda f)(s x_i) = f(s)$$

for all $s \in \Gamma_0$ and let

$$m_0 : \ell^\infty(\Gamma_0) \ni f \longmapsto m(\Lambda f).$$

It is easy to see that m_0 is a state on $\ell^\infty(\Gamma_0)$. Moreover, since for any $t \in \Gamma_0$ we have

$$(\Lambda L_t f)(s x_i) = (L_t f)(s) = f(t^{-1}s) = (\Lambda f)(t^{-1} s x_i) = (L_t \Lambda f)(t),$$

we see that

$$m_0(L_t f) = m(\Lambda L_t f) = m(L_t \Lambda f) = m(L_t \Lambda f) = m_0(f).$$

Ad (2). Let m be a left invariant mean on Γ. We define

$$m_0 : \ell^\infty(\Gamma_0) \ni f \longmapsto m(f \circ \Phi).$$

Clearly m_0 is a state on $\ell^\infty(\Gamma_0)$. Take $x \in \Gamma_0$ and let $y \in \Gamma$ be such that $x = \Phi(y)$. We have

$$\big((L_x f)\circ\Phi\big)(t) = f\big(x^{-1}\Phi(t)\big) = f\big(\Phi(y^{-1}t)\big) = \big(L_y(f\circ\Phi)\big)(t),$$

so that

$$m_0(L_x f) = m\big((L_x f)\circ\Phi\big) = m\big(L_y(f\circ\Phi)\big) = m(f\circ\Phi) = m_0(f).$$

Ad (3). Let m_0 be an invariant mean on $\ell^\infty(\Gamma_0)$ and m_1 an invariant mean on Γ/Γ_0. Let us define a bounded linear map $T\colon \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$ by letting

$$(Tf)(t) = m_0\big((\delta_{t^{-1}} * f)\big|_{\Gamma_0}\big).$$

Now if $t \in \Gamma$ and $s \in \Gamma_0$ then from the left invariance of m_0 we obtain.

$$\begin{aligned}(Tf)(ts) &= m_0\big((\delta_{s^{-1}t^{-1}} * f)\big|_{\Gamma_0}\big) = m_0\big((\delta_{s^{-1}} * \delta_{t^{-1}} * f)\big|_{\Gamma_0}\big)\\ &= m_0\big(\delta_{s^{-1}} * (\delta_{t^{-1}} * f)\big|_{\Gamma_0}\big) = m_0\big((\delta_{t^{-1}} * f)\big|_{\Gamma_0}\big) = (Tf)(t)\end{aligned}$$

In other words Tf is constant on cosets $t\Gamma_0$ of Γ_0. We let $\widetilde{T}f$ denote the function on Γ/Γ_0 such that

$$(Tf)(t) = (\widetilde{T}f)(t\Gamma_0)$$

for all $t \in \Gamma$. This defines a bounded linear map $\widetilde{T}\colon \ell^\infty(\Gamma) \to \ell^\infty(\Gamma/\Gamma_0)$ preserving unit and positivity. Let $m = m_1 \circ \widetilde{T}$.

To check left invariance of m we first see that for $r \in \Gamma$

$$\big(T(L_r f)\big)(t) = m_0\big((\delta_{t^{-1}} * \delta_r * f)\big|_{\Gamma_0}\big) = m_0\big((\delta_{(r^{-1}t)^{-1}} * f)\big|_{\Gamma_0}\big) = (Tf)(r^{-1}t)$$

for all t. Therefore

$$\big(\widetilde{T}(L_r f)\big)(t\Gamma_0) = \big(T(L_r f)\big)(t) = (Tf)(r^{-1}t) = (\widetilde{T}f)(r^{-1}t\Gamma_0) = (\widetilde{T}f)(r^{-1}\Gamma_0 t\Gamma_0)$$

(this is the product in the quotient group Γ/Γ_0). Now m_1 is left invariant, so

$$m(L_r f) = m_1\big(\widetilde{T}(L_r f)\big) = m_1\big(L_{r^{-1}\Gamma_0}\widetilde{T}f\big) = m_1(\widetilde{T}f) = m(f).$$

Ad (4). For each α let m_α be the mean on Γ given by first restricting a function to the subgroup Γ_α and the applying the mean on Γ_α. This way (m_α) becomes a net of norm one functionals on $\ell^\infty(\Gamma)$ all of which map positive elements to positive elements (and have value 1 on the constant functions 1 — this actually follows from the first two conditions) or in other words a net of states. Let m be a weak* accumulation point of this net. Then clearly m is a state on $\ell^\infty(\Gamma)$. To see that m is left invariant we note that if $t \in \Gamma$ then there exists α_0 such that $t \in \Gamma_\alpha$ for all $\alpha \succcurlyeq \alpha_0$. Therefore for any $\alpha \succcurlyeq \alpha_0$ we have $m_\alpha(L_t f) = m_\alpha(f)$ because m_α is invariant under left multiplication by elements of Γ_α. It follows easily that $m(L_t f) = m(f)$ for any t and any $f \in \ell^\infty(\Gamma)$. ☐

7. Day's conditions

We will denote by $\ell^1_+(\Gamma)_1$ the set of norm one positive functions in $\ell^1(\Gamma)$. As functionals on $\ell^\infty(\Gamma)$ these are states (more precisely all states arising from elements of $\ell^1(\Gamma)$ are necessarily given by elements of $\ell^1_+(\Gamma)_1$). For $\phi \in \ell^1(\Gamma)$ we will denote the value of the functional on $\ell^\infty(\Gamma)$ corresponding to ϕ on $f \in \ell^\infty(\Gamma)$ by

$$\langle \phi, f \rangle = \sum_{s \in \Gamma} \phi(s) f(s).$$

Proposition 7.1. *$\ell^1_+(\Gamma)_1$ as a subset of $\ell^\infty(\Gamma)^*$ is weak* dense in $\mathcal{S}\bigl(\ell^\infty(\Gamma)\bigr)$.*

Proof. Let $\mathcal{S}_0$ be the image of $\ell^1_+(\Gamma)_1$ in $\mathcal{S}\bigl(\ell^\infty(\Gamma)\bigr)^*$. Clearly $\mathcal{S}_0$ is a convex set. If ω is a state on ℓ^∞ which is not a limit of a net of elements of $\mathcal{S}_0$ then there exists an element $f \in \ell^\infty(\Gamma)$ and $\delta > 0$ such that

$$\Re\,\omega(f) \geq \delta + \Re\,\eta(f)$$

for all $\eta \in \mathcal{S}_0$. Note that $\mathcal{S}_0$ contains all evaluation functionals and so we have

$$\omega(\Re f) = \Re\,\omega(f) \geq \delta + \Re f(t)$$

for all $t \in \Gamma$. Therefore $\omega(\Re f) > \sup \Re f$ which is impossible for a state. ☐

Definition 7.2. We say that Γ satisfies *weak Day's condition* if there exists a net (ϕ_i) of elements of $\ell^1_+(\Gamma)_1$ such that for any $t \in \Gamma$

$$L_t\phi_i - \phi_i \xrightarrow{\text{weak}^*} 0$$

in $\ell^\infty(\Gamma)^*$. The group Γ satisfies *strong Day's condition* if there exists a net (ϕ_i) as above such that

$$\|L_t\phi_i - \phi_i\|_1 \longrightarrow 0.$$

We are speaking of weak* topology on $\ell^1(\Gamma)$ embedded in the dual of $\ell^\infty(\Gamma)$, i.e. in its bi-dual. This means simply the *weak topology* on $\ell^1(\Gamma)$. The only reason for this strange approach is that we want to treat elements of $\ell^1_+(\Gamma)_1$ as states on $\ell^\infty(\Gamma)$ and the natural topology on the state space is the weak* topology.

Theorem 7.3. *Γ satisfies strong Day's condition if and only if Γ satisfies weak Day's condition.*

Proof. Let E be the product of $|\Gamma|$ copies of $\ell^1(\Gamma)$:

$$E = \prod_{t \in \Gamma} \ell^1(\Gamma).$$

Then E is a locally convex topological vector space with seminorms $\bigl\{\|\cdot\|_t\bigr\}_{t\in\Gamma}$ given by

$$\|F\|_t = \|F_t\|_1,$$

where $F = (F_t)_{t\in\Gamma} \in E$. Define $T : \ell^1(\Gamma) \to E$

$$(Tf)_t = L_t f - f.$$

Let (ϕ_i) be the net of elements of $\ell^1_+(\Gamma)_1$ such that

$$L_t\phi_i - \phi_i \xrightarrow{\text{weak}^*} 0$$

for all $t \in \Gamma$ (weak* topology taken from $\ell^\infty(\Gamma)^*$). Then $T\phi_i \longrightarrow 0$ in E in the weak topology (because this is the product of weak topologies, i.e. a net $(F^\alpha)_{\alpha\in A}$ converges to 0 weakly in E if and only if for any t $\bigl(F^\alpha_t\bigr)_{\alpha\in A}$ converges weakly to 0 in $\ell^1(\Gamma)$. In other words zero is in the weak closure of $T\big(\ell^1_+(\Gamma)_1\big)$. But, as $T\big(\ell^1_+(\Gamma)_1\big)$ is a convex set, its weak and strong closures coincide, so there is a net (ϕ_j) in $\ell^1_+(\Gamma)_1$ such that

$$T\phi_j \xrightarrow{\text{strong}} 0$$

in E, i.e. for any $t \in \Gamma$

$$\|L_t\phi_j - \phi_j\|_1 \longrightarrow 0.$$

□

Theorem 7.4. Γ *is amenable if and only if* Γ *satisfies weak Day's condition.*

Proof. Assume that Γ satisfies weak Day's condition, i.e. there is a net (ϕ_i) of elements of $\ell^1_+(\Gamma)_1$ such that

$$L_t\phi_i - \phi_i \xrightarrow{\text{weak}^*} 0$$

for any t. This net has an accumulation point in the weak* compact set of all states of $\ell^\infty(\Gamma)$. Let m be such an accumulation point and let us use the same notation (ϕ_i) for the subnet weak* convergent to m. We have for any $f \in \ell^\infty(\Gamma)$ and $t \in \Gamma$

$$\begin{aligned} m(L_tf) - m(f) \longleftarrow \langle\phi_i, L_tf\rangle - \langle\phi_i, f\rangle &= \sum_{s\in\Gamma}\big(f(t^{-1}s) - f(s)\big)\phi_i(s) \\ &= \sum_{s\in\Gamma}\big(\phi_i(ts) - \phi_i(s)\big)f(s) \\ &= \langle L_{t^{-1}}\phi_i - \phi_i, f\rangle \longrightarrow 0. \end{aligned}$$

so m is a left invariant mean and Γ is amenable.

Now assume that Γ is amenable and m is a left invariant mean on $\ell^\infty(\Gamma)$. By the weak* density of states arising from elements of $\ell^1_+(\Gamma)_1$ in the state space of $\ell^\infty(\Gamma)$ we can find a net (ϕ_i) of elements of $\ell^1_+(\Gamma)_1$ such that the corresponding states converge to m in the weak* topology. Therefore for any $f \in \ell^\infty(\Gamma)$ and $t \in \Gamma$ we have

$$\begin{aligned} \langle L_t\phi_i - \phi_i, f\rangle &= \langle L_t\phi_i, f\rangle - \langle\phi_i, f\rangle \\ &= \langle\phi_i, L_{t^{-1}}f\rangle - \langle\phi_i, f\rangle \\ &= \langle\phi_i, L_{t^{-1}}f\rangle - m(f) + m(f) - \langle\phi_i, f\rangle \\ &= \langle\phi_i, L_{t^{-1}}f\rangle - m(L_{t^{-1}}f) + m(f) - \langle\phi_i, f\rangle \longrightarrow 0 + 0. \end{aligned}$$

This shows that $L_t\phi_i - \phi_i$ converges to 0 in the weak* topology. □

8. Reiter's condition

Definition 8.1. We say that Γ satisfies *Reiter's condition* if for any $\varepsilon > 0$ and any finite $F \subset \Gamma$ there exists $\phi \in \ell^1_+(\Gamma)_1$ such that

$$\|L_s\phi - \phi\|_1 < \varepsilon \tag{3}$$

for all $s \in F$.

Theorem 8.2. Γ *is amenable if and only if* Γ *satisfies Reiter's condition.*

Proof. Assume that Reiter's condition holds. The collection $\mathcal{I}$ of pairs (F, ε), where $F \subset \Gamma$ is finite and $\varepsilon > 0$ is ordered by

$$\Big((F_1, \varepsilon_1) \geq (F_2, \varepsilon_2)\Big) \Longleftrightarrow \Big(F_1 \supset F_2 \quad \text{and} \quad \varepsilon_1 \leq \varepsilon_2\Big).$$

For $(F, \varepsilon) = i \in \mathcal{I}$ let ϕ_i be the function in $\ell^1_+(\Gamma)_1$ such that

$$\|L_s\phi_i - \phi_i\|_1 < \varepsilon$$

for all $s \in F$. It is clear that for any $t \in \Gamma$ we have

$$L_t\phi_i - \phi_i \longrightarrow 0$$

in $\ell^1(\Gamma)$, i.e. strong Day's condition is satisfied.

To prove the converse implication we note that amenability implies weak Day's condition, which by Theorem 7.3 implies strong Day's condition. This means that we have a net (ϕ_i) in $\ell^1_+(\Gamma)_1$ such that for any $t \in \Gamma$

$$\|L_t\phi_i - \phi_i\|_1 \longrightarrow 0.$$

In other words for a fixed $t \in \Gamma$ and $\varepsilon > 0$ there is an index $i_{i,\varepsilon}$ such that for all $i \geq i_{t,\varepsilon}$ we have

$$\|L_t\phi_i - \phi_i\|_1 < \varepsilon.$$

Let a finite $F \subset \Gamma$ and $\varepsilon > 0$ be given. Denote the elements of F by $\{t_1, \ldots, t_n\}$ and let i_ε be such that $i_\varepsilon \geq i_{t_k,\varepsilon}$ for $k = 1, \ldots, n$. Then for any $i \geq i_\varepsilon$ we have

$$\|L_s\phi_i - \phi_i\|_1 < \varepsilon$$

for all $s \in F$. Putting $\phi = \phi_{i_\varepsilon}$ we obtain an element in $\ell^1_+(\Gamma)_1$ such that (3) holds for all $s \in F$. ☐

8.1. **Følner's condition.** Let us briefly recall a related condition on Γ describing amenability.

Definition 8.3. We say that Γ satisfies Følner's condition if for any finite subset F of Γ and any $\varepsilon > 0$ there exists a nonempty and finite $U \subset \Gamma$ such that

$$\frac{|tU \div U|}{|U|} < \varepsilon$$

for any $t \in F$.

We will leave the next theorem without proof.

Theorem 8.4. Γ *is amenable if and only if* Γ *satisfies Følner's condition.*

9. Amenability in terms of positive definite functions

Let $\psi \in \ell^2(\Gamma)$. Let us introduce the notation $\widetilde{\psi}(t) = \psi(t^{-1})$. Then the convolution product $\varphi = \overline{\psi} * \widetilde{\psi}$ is well defined and it is a positive definite function associated with the regular representation:

$$\varphi(t) = \sum_{s\in\Gamma} \overline{\psi(s)}\widetilde{\psi}(s^{-1}t) = \sum_{s\in\Gamma} \overline{\psi(s)}\psi(t^{-1}s) = (\psi|\lambda_t|\psi)\,.$$

Theorem 9.1. Γ *is amenable if and only if for any finite* $F \subset \Gamma$ *and any* $\varepsilon > 0$ *there exists* $\psi \in \ell^2(\Gamma)_1$ *such that*

$$\left|1 - \overline{\psi} * \widetilde{\psi}(t)\right| < \varepsilon$$

for all $t \in F$.

Proof. Assume first that Γ is amenable. Then Reiter's condition is satisfied. This means that for any finite $F \subset \Gamma$ and $\varepsilon > 0$ there is $\phi \in \ell^1_+(\Gamma)_1$ such that

$$\|L_t\phi - \phi\|_1 < \varepsilon^2$$

for all $t \in F$. Let $\psi = \phi^{\frac{1}{2}}$. Then $\psi \in \ell^2(\Gamma)$ and $\psi \geq 0$. We have

$$\|\lambda_t\psi - \psi\|_2^2 = \sum_{s\in\Gamma}\left|\psi(t^{-1}s) - \psi(s)\right|^2 \leq \sum_{s\in\Gamma}\left|\psi(t^{-1}s)^2 - \psi(s)^2\right| = \sum_{s\in\Gamma}\left|\phi(t^{-1}s) - \phi(s)\right| = \|L_t\phi - \phi\| < \varepsilon^2$$

for all $t \in F$.[iii] Therefore for those t we have $\|\lambda_t\psi - \psi\|_2 < \varepsilon$. Furthermore, since $\|\psi\|_2^2 = 1$ we can compute for $t \in F$

$$\begin{aligned}
\left|1 - (\overline{\psi} * \widetilde{\psi})(t)\right| &= \left|(\psi|\psi) - (\overline{\psi} * \widetilde{\psi})(t)\right| \\
&= \left|(\psi|\psi) - (\psi|\lambda_t|\psi)\right| \\
&= \left|(\psi|\lambda_t\psi - \psi)\right| \leq \|\psi\|_2\|\lambda_t\psi - \psi\|_2 < \varepsilon.
\end{aligned}$$

Conversely, assume now that for any finite $K \subset \Gamma$ and any $\delta > 0$ there exists $\psi \in \ell^2(\Gamma)_1$ such that

$$\left|1 - (\overline{\psi} * \widetilde{\psi})(t)\right| < \delta \tag{4}$$

[iii] For numbers $a, b \geq 0$ we have $(a - b)^2 \leq |a^2 - b^2|$.

for all $t \in K$. Let us fix finite $F \subset \Gamma$ and $\varepsilon > 0$. Let $K = F \cup F^{-1}$ and $\delta = \frac{\varepsilon^2}{8}$. Then, by assumption, there exists $\psi \in \ell^2(\Gamma)_1$ such that

$$\left|1 - (\overline{\psi} * \widetilde{\psi})(t)\right| < \tfrac{\varepsilon^2}{8}$$

for any $t \in F \cup F^{-1}$. Put $\phi = |\psi|^2$. Then $\phi \in \ell^1_+(\Gamma)_1$. We have for $t \in F$

$$\begin{aligned}
\|L_t\phi - \phi\|_1 &= \sum_{s\in\Gamma}\left||\psi(t^{-1}s)|^2 - |\psi(s)|^2\right| \le \sum_{s\in\Gamma}\left|\psi(t^{-1}s)^2 - \psi(s)^2\right| \\
&= \sum_{s\in\Gamma}\left|\psi(t^{-1}s) + \psi(s)\right|\left|\psi(t^{-1}s) - \psi(s)\right| \\
&= (|\lambda_t\psi + \psi|\,|\,|\lambda_t\psi + \psi|) \\
&= \left|(|\lambda_t\psi + \psi|\,|\,|\lambda_t\psi + \psi|)\right| \\
&\le \|\lambda_t\psi + \psi\|_2\|\lambda_t\psi - \psi\|_2 \\
&\le \big(\|\lambda_t\psi\|_2 + \|\psi\|_2\big)\|\lambda_t\psi - \psi\|_2 \\
&\le 2\|\lambda_t\psi - \psi\|_2 \\
&= 2\,(\lambda_t\psi - \psi\,|\,\lambda_t\psi - \psi)^{\frac{1}{2}} \\
&= 2\big[(\lambda_t\psi|\lambda_1\psi) - (\psi|\lambda_t\psi) - (\lambda_t\psi|\psi) + (\psi|\psi)\big]^{\frac{1}{2}} \\
&= 2\big[\big(1 - (\overline{\psi} * \widetilde{\psi})(t)\big) + \big(1 - (\overline{\psi} * \widetilde{\psi})(t^{-1})\big)\big]^{\frac{1}{2}} \\
&< 2\big(\tfrac{\varepsilon^2}{8} + \tfrac{\varepsilon^2}{8}\big)^{\frac{1}{2}} = \varepsilon
\end{aligned}$$

which proves that Reiter's condition is satisfied. ☐

Corollary 9.2. *Γ is amenable if and only if for any finite $F \subset \Gamma$ and any $\varepsilon > 0$ there is a positive definite function φ associated with the regular representation such that $\varphi(e) = 1$ and*

$$\left|1 - \varphi(t)\right| < \varepsilon$$

for all $t \in F$.

10. Finite supports

Lemma 10.1. *Let φ be a positive definite function on Γ with finite support. Then there is a function $\psi \in \ell^2(\Gamma)$ such that $\varphi = \overline{\psi} * \widetilde{\psi}$. In particular φ is associated with the regular representation of Γ.*

Proof. Let T be the operator $\ell^2(\Gamma) \to \ell^2(\Gamma)$ defined as

$$\phi \longmapsto \phi * \overline{\varphi}.$$

Then T is a bounded operator (right convolution by a summable function is bounded and JTJ is the left convolution by $\widetilde{\varphi}$, where J is the antilinear operator $(J\phi)(s) = \overline{\phi(s^{-1})}$ for $\phi \in \ell^2(\Gamma)$ and all $s \in \Gamma$). Moreover T commutes with operators $\{\lambda_s\}_{s\in\Gamma}$: for any $\phi \in \ell^2(\Gamma)$ and any $s, t \in \Gamma$

$$T(\lambda_s\phi) = (\lambda_t\phi) * \overline{\varphi} = (\delta_t * \phi) * \overline{\varphi} = \delta_t * (\phi * \overline{\varphi}) = \lambda_t(T\phi).$$

Let us also see that T is a positive operator: for any ϕ with finite support

$$\begin{aligned}(\phi|T|\phi) &= (\phi|\phi * \overline{\varphi})\\ &= \sum_{s\in\Gamma}\overline{\phi(s)}(\phi*\overline{\varphi})(s)\\ &= \sum_{s\in\Gamma}\overline{\phi(s)}\biggl(\sum_{r\in\Gamma}\phi(r)\overline{\varphi(r^{-1}s)}\biggr)\\ &= \sum_{r,s\in\Gamma}\overline{\phi(s)\varphi(r^{-1}s)}\phi(r)\\ &= \overline{\sum_{r,s\in\operatorname{supp}\phi}\overline{\phi(r)}\varphi(r^{-1}s)\phi(s)} \geq 0.\end{aligned}$$

because φ is positive definite. Clearly also $(\phi|T|\phi)\geq 0$ for any other $\phi\in\ell^2(\Gamma)$ by taking limits.

Now let us take $\psi = T^{\frac{1}{2}}\delta_e$. Then for any $t\in\Gamma$

$$\begin{aligned}\bigl(\overline{\psi}*\widetilde{\psi}\bigr)(t) = (\psi|\lambda_t|\psi) &= \Bigl(T^{\frac{1}{2}}\delta_e\Big|\lambda_t\Big|T^{\frac{1}{2}}\delta_e\Bigr)\\ &= \Bigl(T^{\frac{1}{2}}\delta_e\Big|\lambda_t T^{\frac{1}{2}}\Big|\delta_e\Bigr)\\ &= \Bigl(T^{\frac{1}{2}}\delta_e\Big|T^{\frac{1}{2}}\lambda_t\Big|\delta_e\Bigr)\\ &= (T\delta_e|\lambda_t|\delta_e) = (\overline{\varphi}|\lambda_t\delta_e) = (\overline{\varphi}|\delta_t) = \varphi(t).\end{aligned}$$

□

In the proof of the next theorem we will use the following version of Reiter's condition: for any finite $F\subset\Gamma$ and $\varepsilon>0$ there exists $\phi\in\ell^1_+(\Gamma)_1$ with finite support such that

$$\|L_t\phi-\phi\|_1<\varepsilon$$

for all $t\in F$. This is satisfied by Γ is and only if Reiter's condition holds in its original formulation. Indeed, if F and ε are given and $\psi\in\ell^1_+(\Gamma)_1$ is such that

$$\|L_t\psi-\psi\|_1<\tfrac{\varepsilon}{3}$$

for all $t\in F$. There is a finitely supported $\phi\in\ell^1_+(\Gamma)_1$ such that

$$\|\phi-\psi\|_1<\tfrac{\varepsilon}{3}.$$

To construct such a ϕ find a finite $S\subset\Gamma$ such that $\|\psi\cdot\chi_{\Gamma\setminus S}\|_1<\frac{\varepsilon}{6}$. Put $\phi=\psi$ on S and add value $\alpha=1-\|\psi\cdot\chi_S\|_1=\|\psi\cdot\chi_{\Gamma\setminus S}\|_1$ at some point $s_0\notin S$ and set $\phi=0$ elsewhere. Then $\|\phi\|_1=1$ and ϕ has finite support by construction. Clearly $\alpha\geq\psi(s_0)$. Moreover $\alpha<\frac{\varepsilon}{6}$ and

$$\|\phi-\psi\|_1=\sum_{t\notin(S\cup\{s_0\})}\psi(t)+\alpha-\psi(s_0)<\tfrac{\varepsilon}{6}+\tfrac{\varepsilon}{6}.$$

Then we have for any $t\in F$

$$\begin{aligned}\|L_t\phi-\phi\|_1 &\leq \|L_t\phi-L_t\psi\|_1+\|L_t\psi-\psi\|_1+\|\psi-\phi\|_1\\ &\leq \|L_t\|\|\phi-\psi\|_1+\|L_t\psi-\psi\|_1+\|\psi-\phi\|_1<\varepsilon,\end{aligned}$$

since $\|L_t\|=1$ (as an operator on $\ell^1(\Gamma)$). This shows that Reiter's condition in the original formulation implies the stronger version. The converse implication is obvious.

Theorem 10.2. *Γ is amenable if and only if for any finite $F\subset\Gamma$ and any $\varepsilon>0$ there is a finitely supported positive definite function φ such that $\varphi(e)=1$ and*

$$\bigl|1-\varphi(t)\bigr|<\varepsilon$$

for all $t\in F$.

Proof. The "if" part follows from combination of Lemma 10.1 and Corollary 9.2.

Now if we assume that Γ is amenable then Reiter's condition holds and thus by the discussion preceding formulation of our theorem we have for any finite $F \subset \Gamma$ and $\varepsilon > 0$ a finitely supported $\phi \in \ell^1_+(\Gamma)_1$ such that

$$\|L_t\phi - \phi\|_1 < \varepsilon$$

for all $t \in F$. Arguing in exactly the same way as in the first part of the proof of Theorem 9.1 we find $\psi \in \ell^2(\Gamma)$ with finite support (namely $\psi = \phi^{\frac{1}{2}}$) such that the positive definite function

$$\varphi = \overline{\psi} * \widetilde{\psi}$$

satisfies

$$\big|1 - \varphi(t)\big| < \varepsilon$$

for all $t \in F$. Clearly φ has finite support. □

11. Reduced group C*-algebra and injectivity of the regular representation

As we saw in Section 1 the regular representation $\lambda : \Gamma \to \mathrm{B}\big(\ell^2(\Gamma)\big)$ is faithful on $\mathbb{C}[\Gamma]$, so that the function

$$\mathbb{C}[\Gamma] \ni f \longmapsto \|\lambda_f\|_{\mathrm{B}(\ell^2(\Gamma))}$$

is a C*-norm. The completion of $\mathbb{C}[\Gamma]$ with respect to this norm is the *reduced* C*-*algebra* of Γ denoted by $\mathrm{C}^*_r(\Gamma)$. Clearly the identity map $\mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ extends uniquely to a norm decreasing map $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*_r(\Gamma)$. This map is surjective since it has dense image and the image of a C*-algebra under a $*$-homomorphism is always closed. We will denote this map also by the symbol λ since it is clearly the representation of $\mathrm{C}^*(\Gamma)$ associated with the regular representation of Γ.

Note that for each $t \in \Gamma$ the function $\delta_t \in \mathbb{C}[\Gamma]$ corresponds to the operator $\lambda_t \in \mathrm{C}^*_r(\Gamma)$.

Theorem 11.1. Γ *is amenable if and only if* $\lambda\colon \mathrm{C}^*(\Gamma) \to \mathrm{C}^*_r(\Gamma)$ *is injective.*

Clearly λ is injective if and only if it is an isomorphism.

Proof of theorem 11.1. Assume first that Γ is amenable. Take $b \in \mathrm{C}^*(\Gamma)$ such that $\lambda(b) = 0$ and let $a = b^*b$. There exists a state ω of $\mathrm{C}^*(\Gamma)$ such that $\omega(a) = \|a\|$. Let $\varphi = \varphi_\omega$ (the positive definite function associated with ω) and let (φ_i) be a net of finitely supported positive definite functions with value 1 at e approximating pointwise the constant function 1. Then $(\varphi\varphi_i)$ approximates the function φ (pointwise). Let $\omega_i = \omega_{\varphi\varphi_i}$ for all i. Then, by Proposition 2 the states (ω_i) converge to ω in the weak* topology.

Since $\varphi\varphi_i$ is finitely supported, there is an element $\psi_i \in \ell^2(\Gamma)$ such that

$$\varphi\varphi_i = \overline{\psi_i} * \widetilde{\psi}_i,$$

so that

$$\omega(\delta_t) = (\varphi\varphi_i)(t) = (\psi_i|\lambda_t|\psi_i)\,.$$

Thus for $f \in \ell^1(\Gamma)$ we have

$$\omega(f) = (\psi_i|\lambda_f|\psi_i)$$

and consequently for $x \in \mathrm{C}^*(\Gamma)$

$$\omega(x) = (\psi_i|\lambda(x)|\psi_i)$$

In particular

$$\omega(a) = \lim \omega_i(a) = \lim\,(\psi_i|\lambda(a)|\psi_i) = 0$$

since $\lambda(a) = 0$.

Now let us assume that λ is injective. Then, as we pointed out before the proof, it is an isomorphism. Moreover, this means that the representation of $\mathrm{C}^*(\Gamma)$ arising from the trivial representation is continuous on $\mathrm{C}^*_r(\Gamma)$. In other words the map

$$\lambda_t \longmapsto 1$$

extends to a continuous map $\epsilon : \mathrm{C}^*_r(\Gamma) \to \mathbb{C}$ (the trivial representation is continuous on $\mathrm{C}^*(\Gamma)$ and we pre-compose it with the inverse of λ to get ϵ). This is a representation (character), so in particular, a state. Now any state on $\mathrm{C}^*_r(\Gamma)$ is a weak* limit of convex combinations of vector states ($\mathrm{C}_r^*(\Gamma) \subset \mathrm{B}\big(\ell^2(\Gamma)\big)$). Therefore it is a weak* limit of vector states arising from finitely supported functions. Therefore There is a net of states (ω_i) on $\mathrm{C}_r^*(\Gamma)$ weak* convergent to ϵ such that the positive definite functions $(\varphi_{\omega_i \circ \lambda})$ have finite support. It follows that the constant function 1, i.e. the positive definite function corresponding to ϵ can be pointwise approximated by positive definite functions of finite support taking value 1 at e. By Theorem 10.2 the group Γ is amenable. ☐

12. Completely positive maps

Let A and B be C*-algebras.

Definition 12.1. Let $\Phi : A \to B$ be a linear map. We say that Φ is positive if it takes positive elements to positive elements. For any $n \in \mathbb{N}$ we define $\Phi_n : M_n(A) \to M_n(B)$ given by

$$\Phi_n \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = \begin{bmatrix} \Phi(a_{1,1}) & \Phi(a_{1,2}) & \cdots & \Phi(a_{1,n}) \\ \Phi(a_{2,1}) & \Phi(a_{2,2}) & \cdots & \Phi(a_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(a_{n,1}) & \Phi(a_{n,2}) & \cdots & \Phi(a_{n,n}) \end{bmatrix}.$$

We say that Φ is completely positive (c.p.) if Φ_n is positive for all n.

We will sometimes use the abbreviation “u.c.p.” for “unital completely positive”.

Lemma 12.2. *A linear map $\Phi : A \to B$ is completely positive if and only if for any $n \in \mathbb{N}$ and any $a_1, \ldots, a_n \in A$ the matrix*

$$\begin{bmatrix} \Phi(a_1^*a_1) & \Phi(a_1^*a_2) & \cdots & \Phi(a_1^*a_n) \\ \Phi(a_2^*a_1) & \Phi(a_2^*a_2) & \cdots & \Phi(a_2^*a_n) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(a_n^*a_1) & \Phi(a_n^*a_2) & \cdots & \Phi(a_n^*a_n) \end{bmatrix} \in M_n(B) \tag{5}$$

is positive.

Proof. If Φ is c.p. then (5) is positive because the matrix

$$\begin{aligned} \begin{bmatrix} a_1^*a_1 & a_1^*a_2 & \cdots & a_1^*a_n \\ a_2^*a_1 & a_2^*a_2 & \cdots & a_2^*a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^*a_1 & a_n^*a_2 & \cdots & a_n^*a_n \end{bmatrix} &= \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \\ &= \begin{bmatrix} a_1^* & 0 & \cdots & 0 \\ a_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n^* & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{6}$$

is positive.

Conversely we can show that any matrix

$$a = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \in M_n(A)_+$$

is a finite sum of matrices of the form (6) which proves the lemma.

So let $a \in M_n(A)$ be positive. There exists a matrix

$$b = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \in M_n(A)$$

such that $a = b^*b$ which means that $a_{i,j} = \sum\limits_{k=1}^{n} b^*_{k,i} b_{k,j}$. In other words if

$$c_k = \begin{bmatrix} b^*_{k,1}b_{k,1} & b^*_{k,1}b_{k,2} & \cdots & b^*_{k,1}b_{k,n} \\ b^*_{k,2}b_{k,1} & b^*_{k,2}b_{k,2} & \cdots & b^*_{k,2}b_{k,n} \\ \vdots & \vdots & \ddots & \vdots \\ b^*_{k,n}b_{k,1} & b^*_{k,n}b_{k,2} & \cdots & b^*_{k,n}b_{k,n} \end{bmatrix} = \begin{bmatrix} b^*_{k,1} \\ b^*_{k,2} \\ \vdots \\ b^*_{k,n} \end{bmatrix} \begin{bmatrix} b_{k,1} & b_{k,2} & \cdots & b_{k,n} \end{bmatrix}$$

then we have $a = \sum\limits_{k=1}^{n} c_k$. □

Corollary 12.3. *Let $B \subset \mathrm{B}(H)$. Then $M_n(B) \subset \mathrm{B}(\mathbb{C}^n \otimes H)$ and we have that $\Phi : A \to B$ is c.p. if and only if for any $n \in \mathbb{N}$, any $a_1, \ldots, a_n \in A$ and any $\xi_1, \ldots, \xi_n \in H$ we have*

$$\sum_{i,j=1}^{n} (\xi_i | \Phi(a_i^* a_j) | \xi_j) \geq 0.$$

Example 12.4. Let A be a C*-algebra and let H and K be Hilbert spaces. Let also $\pi\colon A \to \mathrm{B}(K)$ be a $*$-representation and let $T \in \mathrm{B}(H, K)$. Then the map

$$\Phi\colon A \ni a \longmapsto T^*\pi(a)T \in \mathrm{B}(H)$$

is completely positive. Indeed, $\Phi_n\colon M_n(A) \to M_n\bigl(\mathrm{B}(H)\bigr)$ is a composition of

$$M_n(A) \ni \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \longmapsto \begin{bmatrix} \pi(a_{1,1}) & \pi(a_{1,2}) & \cdots & \pi(a_{1,n}) \\ \pi(a_{2,1}) & \pi(a_{2,2}) & \cdots & \pi(a_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ \pi(a_{n,1}) & \pi(a_{n,2}) & \cdots & \pi(a_{n,n}) \end{bmatrix} \in M_n\bigl(\mathrm{B}(K)\bigr)$$

(which is a $*$-homomorphism and hence is positive) and the map

$$M_n\bigl(\mathrm{B}(K)\bigr) = \mathrm{B}(\mathbb{C}^n \otimes K) \ni X \longmapsto (\mathbb{1} \otimes T)^* X (\mathbb{1} \otimes T) \in \mathrm{B}(\mathbb{C}^n \otimes H) = M_n\bigl(\mathrm{B}(H)\bigr)$$

which obviously preserves positivity.

The *Stinespring theorem* says that any c.p. map $A \to \mathrm{B}(H)$ has the form described in Example 12.4.

13. Completely positive maps and positive definite functions

Let $\varphi : \Gamma \to \mathbb{C}$ be a positive definite function. Then φ defines a linear map $m_\varphi : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ by pointwise multiplication (we interpret $\mathbb{C}[\Gamma]$ as finitely supported functions on Γ).

Proposition 13.1. *Let $\varphi : \Gamma \to \mathbb{C}$ be a positive definite function. Then $m_\varphi : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ extends uniquely to a c.p. map $\Phi_\varphi : \mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$. If $\varphi(e) = 1$ then Φ_φ is unital.*

Proof. Let us form the GNS triple (H, π, ξ) such that

$$\varphi(t) = (\xi | \pi(\delta_t) | \xi)$$

for all $t \in \Gamma$ (as in Section 3) and let us define $S : \ell^2(\Gamma) \to \ell^2(\Gamma, H)$ by

$$(S\psi)(s) = \psi(s)\pi(\delta_s)^*\xi.$$

S is bounded because

$$\begin{aligned} \|S\psi\|^2 &= \sum_{s \in \Gamma} \bigl\|\psi(s)\pi(\delta_s)^*\xi\bigr\|^2 \\ &= \sum_{s \in \Gamma} \bigl|\psi(s)\bigr|^2 \bigl\|\pi(\delta_s)^*\xi\bigr\|^2 \\ &\leq \|\xi\|^2 \sum_{s \in \Gamma} \bigl|\psi(s)\bigr|^2 = \|\xi\|^2 \|\psi\|_2^2, \end{aligned}$$

so that $\|S\| \leq \|\xi\|$. Moreover for $\Psi \in \ell^2(\Gamma, H)$ and $t \in \Gamma$ we have

$$\begin{aligned}(S^*\Psi)(t) = (\delta_t | S^*\Psi) = (S\delta_t|\Psi) &= \sum_{s\in\Gamma} ((S\delta_t)(s)|\Psi(s)) \\ &= \sum_{s\in\Gamma} (\delta_t(s)\pi(\delta_s)^*\xi|\Psi(s)) \\ &= (\pi(\delta_t)^*\xi|\Psi(t)) = (\xi|\pi(\delta_t)|\Psi(t)).\end{aligned}$$

The space $\ell^2(\Gamma, H)$ is naturally isomorphic to the Hilbert space tensor product $\ell^2(\Gamma) \otimes H$. Therefore for any $t \in \Lambda$ we have the operaotr $\lambda_t \otimes \mathbb{1}_H$ on $\ell^2(\Gamma, H)$. Under the natural isomorphism $\ell^2(\Gamma, H) \to \ell^2(\Gamma) \otimes H$ this operator becomes simply $\big((\lambda_t \otimes \mathbb{1}_H)\Psi\big)(s) = \Psi(t^{-1}s)$.

Now for $x \in \Gamma$ and $\psi \in \ell^2(\Gamma)$ let us compute

$$\begin{aligned}\big(S^*(\lambda_x \otimes \mathbb{1}_H)S\psi\big)(t) &= \big(\xi\big|\pi(\delta_t)\big|\big((\lambda_x \otimes \mathbb{1}_H)S\psi\big)(t)\big) \\ &= \big(\xi\big|\pi(\delta_t)\big|(S\psi)(x^{-1}t)\big) \\ &= \big(\xi\big|\pi(\delta_t)\big|\psi(x^{-1}t)\pi(\delta_{x^{-1}t})^*\xi\big) \\ &= \psi(x^{-1}t)\,(\xi|\pi(\delta_x)|\xi) = \psi(x^{-1}t)\varphi(x).\end{aligned}$$

This shows that $S^*(\lambda_x \otimes \mathbb{1}_H)S = \varphi(x)\lambda_x = m_\varphi(\delta_x)$. Therefore, by linearity, we obtain

$$m_\varphi(a) = S^*(a \otimes \mathbb{1}_H)S$$

for all $a \in \mathbb{C}[\Gamma]$. We see that the unique extension Φ_φ of m_φ to a map $\mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$ is completely positive (cf. Example 12.4). If $\varphi(e) = 1$, then $\Phi_\varphi(\mathbb{1}) = \Phi_\varphi(\delta_e) = \delta_e = \lambda_e = \mathbb{1}$. □

Let $\Phi : \mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$ be a completely positive map. Let us define a function $\varphi_\Phi : \Gamma \to \mathbb{C}$

$$\varphi_\Phi(t) = (\delta_e|\Phi(\lambda_t)\lambda_t^*|\delta_e).$$

Proposition 13.2. *Let $\Phi : \mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$ be a completely positive map. Then the function φ_Φ is positive definite. If Φ is unital then $\varphi_\Phi(e) = 1$.*

Proof. Let ρ denote the *right regular representation* of Γ:

$$(\rho_t\psi)(s) = \psi(st)$$

for all $\psi \in \ell^2(\Gamma)$ and $s \in \Gamma$. The operators $\{\rho_t \,|\, t \in \Gamma\}$ commute with $\mathrm{C}^*_r(\Gamma)$. Moreover for any $t \in \Gamma$ we have $\lambda_t^*\delta_e = \rho_t\delta_e$.

Take $t_1, \ldots, t_n \in \Gamma$. We have

$$\begin{aligned}\varphi(t_i^{-1}t_j) &= \Big(\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\lambda^*_{t_i^{-1}t_j}\Big|\delta_e\Big) = \Big(\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\Big|\lambda^*_{t_i^{-1}t_j}\delta_e\Big) \\ &= \Big(\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\Big|\rho_{t_i^{-1}t_j}\delta_e\Big) = \Big(\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\Big|\rho^*_{t_i}\rho_{t_j}\delta_e\Big) \\ &= \Big(\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\rho^*_{t_i}\Big|\rho_{t_j}\delta_e\Big) = \Big(\delta_e\Big|\rho^*_{t_i}\Phi(\lambda_{t_i^{-1}t_j})\Big|\rho_{t_j}\delta_e\Big) \\ &= \Big(\rho_{t_i}\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\Big|\rho_{t_j}\delta_e\Big) = \Big(\rho_{t_i}\delta_e\Big|\Phi(\lambda_{t_i^{-1}t_j})\Big|\rho_{t_j}\delta_e\Big) \\ &= \big(\rho_{t_i}\delta_e\big|\Phi(\lambda^*_{t_i}\lambda_{t_j})\big|\rho_{t_j}\delta_e\big) = \big(\delta_{t_i}\big|\Phi(\lambda^*_{t_i}\lambda_{t_j})\big|\delta_{t_j}\big),\end{aligned}$$

so that the matrix

$$\begin{bmatrix}\varphi(t_1^{-1}t_1) & \varphi(t_1^{-1}t_2) & \cdots & \varphi(t_1^{-1}t_n) \\ \varphi(t_2^{-1}t_1) & \varphi(t_2^{-1}t_2) & \cdots & \varphi(t_2^{-1}t_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(t_n^{-1}t_1) & \varphi(t_n^{-1}t_2) & \cdots & \varphi(t_n^{-1}t_n)\end{bmatrix}$$

is positive by Corollary 12.3. □

Remark 13.3.

(1) Let φ be a finitely supported positive definite function on Γ, then the associated c.p. map Φ_φ maps into a finite dimensional subspace of $\mathrm{C}^*_r(\Gamma)$. Moreover if $\varphi(e) = 1$ then Φ_φ is a unital map.

(2) The mapping $\varphi \mapsto \Phi_\varphi$ is an injection from the set of positive definite functions on Γ into the set of c.p. maps $\mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$, but its far from being a surjection. In fact we have for any $t \in \Gamma$

$$\begin{aligned}\varphi_{\Phi_\varphi}(t) &= (\delta_e | \Phi_\varphi(\lambda_t)\lambda_t^* | \delta_e)\\ &= (\delta_e | \varphi(t)\lambda_t\lambda_t^* | \delta_e)\\ &= \varphi(t)\,(\delta_e|\delta_e) = \varphi(t).\end{aligned}$$

(3) If $\Phi\colon \mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$ is a c.p. map which is *finite dimensional* (has finite dimensional range) then the corresponding positive definite function φ_Φ has finite support.

14. Nuclearity and amenability

Definition 14.1. Let A and B be C^*-algebras.

(1) A u.c.p. map $\Phi : A \to B$ is called *nuclear* if there exists a net (Φ_i) of finite dimensional u.c.p. maps $A \to B$ such that

$$\big\|\Phi_i(a) - \Phi(a)\big\| \longrightarrow 0$$

for any $a \in A$.

(2) A is called *nuclear* if id_A is a nuclear map.

The above definition of nuclearity of a C^*-algebra is not the most common one. In fact the most important aspect of nuclearity has to do with tensor products of C^*-algebras. However, the definition we gave is most suitable for the study of amenability of discrete groups.

Theorem 14.2. *Γ is amenable if and only if $\mathrm{C}^*_r(\Gamma)$ is a nuclear C^*-algebra.*

Proof. Assume that Γ is amenable. Then for any finite $F \subset \Gamma$ and $\varepsilon > 0$ there is a finitely supported positive definite function $\varphi_{F,\varepsilon}$ such that $\varphi_{F,\varepsilon}(e) = 1$ and

$$\big|1 - \varphi_{F,\varepsilon}(t)\big| < \varepsilon$$

for all $t \in F$.

The u.c.p. maps Φ_{φ_F} are all finite dimensional and clearly for any $a \in \mathbb{C}[\Gamma]$ the net $\big(\Phi_{\varphi_{F,\varepsilon}}(a)\big)$ converges to a (the set of all pairs (F,ε) is directed in the obvious way). A standard argument (using the fact that $\|\Phi_{\varphi_{F,\varepsilon}}\| = 1$ for all F and ε) shows that

$$\big\|\Phi_{\varphi_{F,\varepsilon}}(a) - a\big\| \longrightarrow 0$$

for all $a \in \mathrm{C}^*_r(\Gamma)$.

Conversely, let (Φ_i) be a net of u.c.p. maps $\mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma)$ approximating pointwise $\mathrm{id}_{\mathrm{C}^*_r(\Gamma)}$ and consider the net of positive definite functions (φ_{Φ_i}). Since

$$\big\|\Phi_i(\lambda_t) - \lambda_t\big\| \longrightarrow 0$$

in $\mathrm{C}^*_r(\Gamma)$ for any t, we have

$$\varphi_{\Phi_i}(t) = (\delta_e | \Phi_i(\lambda_t)\lambda_t^* | \delta_e) \longrightarrow (\delta_e|\delta_e) = 1.$$

This means that the net (φ_{Φ_i}) approximates the constant function 1 pointwise. By Remark 13.3(3) the functions (φ_{Φ_i}) have finite support and existence of such an approximating net is equivalent to amenability of Γ (Theorem 10.2). $\square$

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